

Parity-breaking electromagnetic interactions in thermal QED₃

F. T. Brandt[†], Ashok Das[‡] and J. Frenkel[†]

[†]Instituto de Física, Universidade de São Paulo
São Paulo, SP 05315-970, BRAZIL

[‡]Department of Physics and Astronomy, University of Rochester
Rochester, NY 14627-0171, USA

February 1, 2008

Abstract

We examine the parity violating terms generated by the *box diagram* in QED₃ at finite temperature. These lead to both *extensive* as well as *non-extensive* effective actions, which have very distinct behavior in the long wavelength and static limits. We discuss a *large gauge Ward identity* for the leading terms in the static limit, whose solution coincides with the effective action proposed earlier.

Three-dimensional gauge theories coupled to matter are of interest in the high-temperature domain of four-dimensional field theories, as well as in the description of *planar phenomena* in condensed matter physics [1]. An important feature of these theories is the parity anomaly which manifests in the generation of a Chern-Simons term in the effective gauge field action, when the fermions are integrated out. The simplest parity-breaking Chern-Simons term appears in QED₃ at the level of the self-energy and has the form

$$I_{CS} = C(T) \int d^3x \epsilon^{\mu\nu\lambda} A_\lambda F_{\mu\nu}, \quad (1)$$

where $C(T)$ is a temperature-dependent coefficient [2]. It is well known that such terms, in a non-Abelian gauge theory, are not invariant under *large gauge* transformations, which are associated with nonzero topological winding numbers [3]. This happens because, at arbitrary temperatures, $C(T)$ cannot be chosen to take discrete values as required by the *large gauge* invariance of the theory. Note that since, at finite temperature, the time direction becomes compact, nontrivial gauge transformations can arise even in an Abelian theory.

Recently much progress has been made in the understanding of *large gauge* invariance at finite temperature. In particular, a mechanism was presented in the context of an exactly soluble $0 + 1$ dimensional model [4], where *large gauge* invariance was noted to be restored in the full effective action. In other words, it was noted, both perturbatively [5] as well as in the exact effective action [4] that although the Chern-Simons term violates *large gauge* invariance, at finite temperature, there arise other terms in the effective action restoring the invariance. An interesting feature of this mechanism, in the $0 + 1$ dimensional finite temperature actions, is the presence of terms which are not simply space-time integrals of a density (the *non-extensive terms*).

These ideas have subsequently been extended to the $2 + 1$ dimensional QED theory, in the special gauge background $A_0 = A_0(t)$ and $A_i = A_i(\vec{x})$, where several aspects of the effective action have been studied [6]. However, there is some reason to believe that the actions obtained in such backgrounds may not represent the complete action. This is because such a choice of gauge backgrounds may be too restrictive. Furthermore, while in such actions only non-extensive terms are present, it is known that in the $1 + 1$ dimensional theory, the finite temperature effective action is extensive (but non-local) [7]. Hence, one would expect such features to appear also in the complete effective action of QED₃, in a general gauge background. Unfortunately, in this case, it is not possible to determine the complete effective action in a closed form (unlike the $0 + 1$ dimensional case). As a result, one has to resort to perturbation theory [5].

The main purpose of this work is to take a first step in this direction, by analyzing, in perturbation theory, the behavior of the parity-breaking part of the thermal electromag-

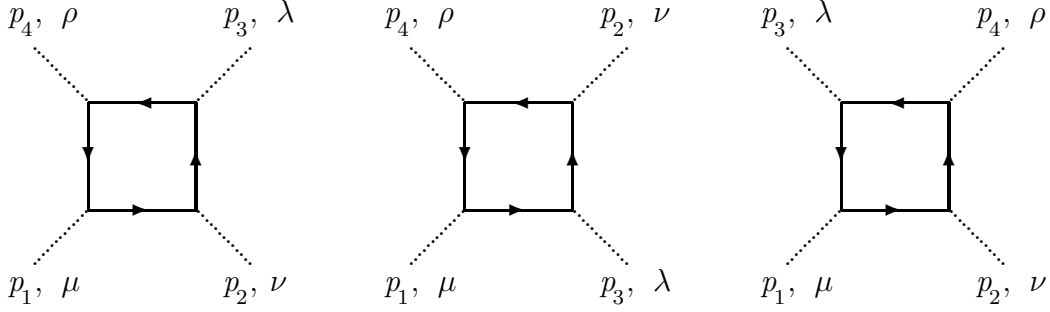


Figure 1: Box diagrams which contribute to the four photon function. Dotted lines represent photons, and solid lines stand for electrons.

netic *four-point function*. We first derive the analytic expression for the corresponding effective action at zero temperature, which has a Lorentz and gauge invariant form, involving only the electromagnetic field tensor. However, at $T \neq 0$, the four-point function is not Lorentz invariant due to the presence of the heat bath, and furthermore, one has to face the issue of non-analyticity of the thermal amplitudes (which is not present in $0 + 1$ dimensions). The calculation of the four-point function, involves a large number of terms in the intermediate steps and is extremely difficult to carry out in general. For this reason, we have studied the finite temperature behavior of the box diagram only in two special, but important limits, namely, in the *long wavelength* and *static* limits. In the long wavelength limit, we find that the thermal contributions give rise to an extensive action which is manifestly invariant under large gauge transformation. These terms have a leading behavior proportional to $1/T$ at high temperature. In contrast, the leading contributions, in the static limit, correspond to a non-extensive action, which is in agreement with the form proposed earlier in the literature [6] and have a leading high temperature behavior proportional to $1/T^3$. There are, in addition extensive terms, which are suppressed at high temperature by extra powers of $1/T$. Large gauge invariance is an important issue in the static limit. In this case, we can write for the leading parity-breaking terms a non linear Ward identity, which reflects the invariance of the full theory under *large gauge* transformations. The solution of such a Ward identity coincides with the non-extensive action proposed earlier in the specific static gauge background. We describe next only the results of our analysis, leaving the details for a separate publication [8].

The graphs which contribute to the four photon function are shown in Fig. 1. There are three other contributions obtained by charge conjugation. To evaluate these dia-

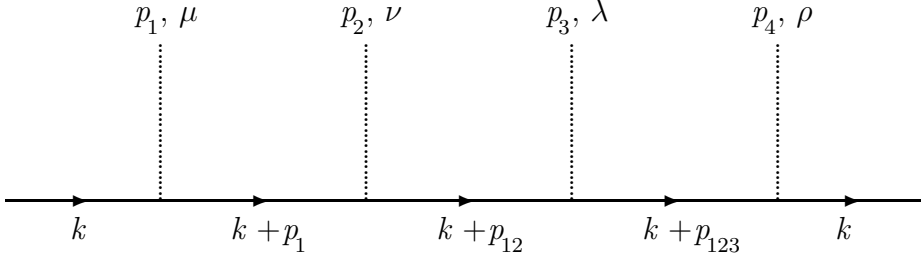


Figure 2: One of the four forward scattering amplitudes, corresponding to the first diagram in Fig. 1.

grams, we use the analytically continued imaginary-time thermal perturbation theory [9, 10, 11]. This approach can be formulated [12] so as to express the thermal Greens function in terms of *forward scattering* amplitudes of an on-shell fermion in an external electromagnetic field, as depicted in Fig. 2. Each of these amplitudes corresponds to a *cut* in one of the internal lines of the boxes in Fig. 1. This generates a total of $4 \times 6 = 24$ *diagrams*, which can be systematically obtained from the graph in Fig. 2, by permutations of the external momenta and polarizations.

The finite-temperature contribution of the box diagrams can then be written in the form

$$\Pi^{\mu\nu\lambda\rho}(p_1, p_2, p_3, p_4) = -\frac{e^4}{(2\pi)^2} \int \frac{d^2\vec{k}}{2\omega_k} \left(\frac{1}{2} - N(\omega_k) \right) \left\{ \sum_{ijkl} B_{(ijkl)}^{\mu\nu\lambda\rho} + (k \leftrightarrow -k) \right\} \quad (2)$$

Here $\omega_k = \sqrt{k^2 + m^2}$, $N(\omega_k) = (e^{\omega_k/T} + 1)^{-1}$, and the sum is over the permutations $(ijkl)$ of (1234) . Each B has a numerator which involves a trace over the Dirac indices. For example

$$B_{(1234)}^{\mu\nu\lambda\rho} = \frac{\text{tr} \left[(\not{k} + m) \gamma^\mu (\not{k} + \not{p}_1 + m) \gamma^\nu (\not{k} + \not{p}_{12} + m) \gamma^\lambda (\not{k} + \not{p}_{123} + m) \gamma^\rho \right]}{(2k \cdot p_1 + p_1^2) (2k \cdot p_{12} + p_{12}^2) (2k \cdot p_{123} + p_{123}^2)} \Big|_{k_0=\omega_k}, \quad (3)$$

where $p_{12} = p_1 + p_2$, etc. Here, we are only interested in the contributions from the trace in Eq. (3) which contain *odd powers of the mass*, since these will lead to parity-breaking terms.

Let us study first the zero temperature contribution, which is associated with the factor $1/2$ in the first bracket of Eq. (2), as $N(\omega_k)$ vanishes in this limit. The computation can be performed explicitly in the *low momentum* region, where $|p_\mu| \ll m$. The

result can then be expressed in terms of a series in powers of p/m , which begins with the leading contribution

$$\begin{aligned} \Pi_{T=0}^{\mu\nu\lambda\rho} &= -\frac{i e^4}{16\pi m^6} \left[\epsilon^{\mu\nu\alpha} p_1^\alpha (p_2)^2 + \epsilon^{\mu\alpha\beta} p_1^\alpha p_2^\beta p_2^\nu \right] \\ &\times \left[\eta^{\lambda\rho} p_3 \cdot p_4 - p_3^\rho p_4^\lambda \right] + \text{permutations} \end{aligned} \quad (4)$$

It is interesting to note that this result is consistent with the Coleman-Hill theorem [13] which implies that, in the four point Greens function, at zero temperature, the terms of order p should be absent. In fact, the above structure shows that the parity-violating contributions, generated by the box at $T = 0$, begin only with terms of order $(p/m)^5$. In the configuration space, the low-energy effective action associated with Eq. (4) can be written in the form

$$\Gamma_{T=0}^4 = -\frac{e^4}{64\pi m^6} \int d^3x \epsilon^{\mu\nu\lambda} F_{\mu\nu} (\partial^\tau F_{\tau\lambda}) F^{\rho\sigma} F_{\rho\sigma}, \quad (5)$$

which is manifestly Lorentz and gauge invariant (*small* and *large*).

We now turn to the parity-violating, finite temperature contributions generated by the box diagram. Such contributions are non-analytic [11], so that the thermal behavior is quite distinct in different regions under consideration. For example, in the *long wavelength* limit, $\vec{p}_i = 0$, these contributions can be expressed as

$$\Pi_{LW}^{\mu\nu\lambda\rho} = i e^4 (\eta^{\mu\nu} - \delta_0^\mu \delta_0^\nu) \epsilon^{0\lambda\rho} (p_3^0 - p_4^0) R_{34}(p_i^0, T) + \text{permutations}, \quad (6)$$

where the functions R_{mn} have, in general, a rather complicated structure depending upon external energies and the temperature.

A great simplification occurs in the low-energy region where $p_i^0 \ll m$, when those functions reduce to a common form given by

$$R = \frac{m T}{64} \sum_{l=-\infty}^{\infty} \left\{ \left(\frac{5 m^2}{\Delta_l^6} + \frac{3}{\Delta_l^4} \right) \ln \left(1 + \frac{\Delta_l^2}{m^2} \right) - \frac{5}{\Delta_l^4} - \frac{1}{2 m^2 \Delta_l^2} \right\}, \quad (7)$$

where $\Delta_l \equiv (2l+1) \pi T$. In the high temperature limit, the leading contribution comes from the last term in Eq. (7). Performing the summation over l , we then obtain that

$$R(T \gg m) = \frac{1}{512} \frac{1}{m T}. \quad (8)$$

It is worth remarking here that the only non-vanishing components of the amplitude in Eq. (6), are the ones when the indices take spatial values. Using this fact, we see that the leading contribution comes from an extensive effective action of the form

$$\tilde{\Gamma}_{LW}^4 = \frac{e^4}{512 m T} \int d^3 x \epsilon_{0ij} E_i \left(\partial_t^{-1} E_j \right) \left(\partial_t^{-1} E_k \right) \left(\partial_t^{-1} E_k \right), \quad (9)$$

where \vec{E} denotes the electric field. This action is non-local and manifestly gauge invariant.

Next, we discuss the thermal behavior of the box in the static limit where $p_i^0 = 0$. In this case, due to the very complicated angular integrations, the calculations are extremely difficult, even when using *computer algebra*. As a result, we have restricted ourselves to a special 3-momentum configuration of the external momenta, where $\vec{p}_1 = \vec{p}_2 = \vec{p}_3 = \vec{p}$ (because of momentum conservation, $\vec{p}_4 = -3\vec{p}$). We then find that the only non-vanishing components of the amplitude are

$$\Pi_S^{000i_4} = \frac{1}{4 |\vec{p}|^2} p_i \epsilon_{0 i i_4} \Pi_1(\vec{p}, T) \quad (10)$$

and,

$$\Pi_S^{0i_2 i_3 i_4} = \frac{1}{12 |\vec{p}|^4} p_i \epsilon_{0 i i_4} \left(p_{i_2} p_{i_3} - |\vec{p}|^2 \delta_{i_2 i_3} \right) \Pi_2(\vec{p}, T), \quad (11)$$

where $\Pi_{1,2}(\vec{p}, T)$ are rather complicated functions of the momenta and the temperature. However, an important simplification occurs in the low momentum region $|\vec{p}| \ll m, T$. In this domain, $\Pi_2 = O(p^6/m^6)$ becomes negligible, while the expression of Π_1 reduces to

$$\tilde{\Pi}_1(\vec{p}, T) = \frac{6 i e^4}{4 \pi} \left[\tanh \left(\frac{m}{2T} \right) - \tanh^3 \left(\frac{m}{2T} \right) \right] \frac{|\vec{p}|^2}{T^2} + O \left(\frac{|\vec{p}|^4}{m^2 T^2} \right) \quad (12)$$

In the high temperature limit, the leading term is of order $1/T^3$, which is quite different from the $1/T$ behavior of the result in the *long wavelength* limit (8).

The leading contribution given in Eqs. (10) and (12) can be associated with the effective non-extensive action

$$\tilde{\Gamma}_S^4 = \frac{e^4}{4 \pi T^2} \left[\tanh \left(\frac{m}{2T} \right) - \tanh^3 \left(\frac{m}{2T} \right) \right] \int d^3 x a^3 B, \quad (13)$$

where

$$a = \int_0^{1/T} dt A_0(t, \vec{x})$$

and $B = B(\vec{x})$ is the static magnetic field. This form is consistent with the result derived from the all-orders effective action, which was obtained earlier in the special gauge background.

In contrast to the effective action obtained in the long wavelength limit [Eq. (9)], the static action (13) *is not invariant* under *large gauge transformations* generated by $a \rightarrow a + 2\pi N$, where N is an integer winding number. (Incidentally, at finite temperature, the effective action is non-unique [14] and depends on the limits in which it is derived.) But one can derive, in this case, a Ward identity for *large gauge* invariance, which relates the amplitudes obtained in perturbation theory. Motivated by the structure of Eq. (13), let us write the all-orders static effective action in the form

$$\tilde{\Gamma}_S = \frac{eT}{2\pi} \int d^3x \tilde{\Gamma}(\tilde{a}) B, \quad (14)$$

where $\tilde{a} = ea$. It has been noted in [5] that in the special background $A_0 = A_0(t)$ and $\vec{A} = \vec{A}(\vec{x})$, $\tilde{\Gamma}(\tilde{a})$ corresponds precisely to the *real part* of the effective action $\Gamma^{(1)}(\tilde{a})$ which describes the behavior of the $0 + 1$ dimensional theory. This action obeys, for a single fermion flavor, the *large gauge Ward identity* [15]

$$\frac{\partial^2 \Gamma^{(1)}}{\partial \tilde{a}^2} = i \left[\frac{1}{4} - \left(\frac{\partial \Gamma^{(1)}}{\partial \tilde{a}} \right)^2 \right], \quad (15)$$

where the one point function has the value

$$\left. \frac{\partial \Gamma^{(1)}}{\partial \tilde{a}} \right|_{\tilde{a}=0} = \frac{1}{2} \tanh \left(\frac{m}{2T} \right). \quad (16)$$

Using these relations, as well as the conditions following from them, we find after some analysis that $\tilde{\Gamma}(\tilde{a}) = \Re [\Gamma^{(1)}(\tilde{a})]$ satisfies the large gauge Ward identity

$$\frac{\partial^2 \tilde{\Gamma}}{\partial \tilde{a}^2} = \frac{1}{\sinh(m/T)} \frac{\partial \tilde{\Gamma}}{\partial \tilde{a}} \sin(2\tilde{\Gamma}), \quad (17)$$

which reflects the large gauge invariance of the static QED₃ theory, in the leading approximation. The nonlinear relation (17), which can be checked perturbatively, shows that all amplitudes are related recursively to the one-point function. The solution of Eq. (17) is given by

$$\tilde{\Gamma}(\tilde{a}) = \arctan \left[\tanh \left(\frac{m}{2T} \right) \tan \left(\frac{\tilde{a}}{2} \right) \right]. \quad (18)$$

Substituting this form in the expression (14), we obtain a result which agrees, in the static limit of QED₃, with the one proposed earlier for the parity-breaking effective action.

This work was supported in part by U.S. Dept. Energy Grant DE-FG 02-91ER40685, NSF-INT-9602559 as well as by CNPq, Brazil.

References

- [1] A nice review of the subject with many references can be found in, G. Dunne, *Topological Aspects of Low Dimensional Systems*, Les Houches Summer School, 1998.
- [2] K. S. Babu, A. Das and P. Panigrahi, Phys. Rev. **D36**, 3725 (1987); E. Poppitz, Phys. Lett. **B252**, 417 (1990); I. J. R. Aitchison, C. Fosco and J. Zuk, Phys. Rev. **D48**, 5895 (1993).
- [3] S. Deser, R. Jackiw and S. Templeton, Ann. Phys. **140**, 372 (1982).
- [4] G. Dunne, K. Lee and C. Lu, Phys. Rev. Lett. **78**, 3434 (1997).
- [5] A. Das and G. Dunne, Phys. Rev. **D57**, 5023 (1998).
- [6] S. Deser, L. Griguolo and D. Seminara, Phys. Rev. Lett. **79**, 1976 (1997); C. Fosco, G. L. Rossini and F. A. Schaposnik, Phys. Rev. Lett. **79**, 1980 (1997) and Phys. Rev. **D56**, 6547 (1997); I. J. R. Aitchison and C. D. Fosco, Phys. Rev. **D57**, 1171 (1998)
- [7] A. Das and A. J. da Silva, Phys. Rev. **D59**, 105011 (1999).
- [8] F. T. Brandt, A. Das and J. Frenkel, in preparation.
- [9] J. I. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1989).
- [10] M. L. Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).
- [11] A. Das, *Finite Temperature Field Theory* (World Scientific, NY, 1997).
- [12] J. Frenkel and J. C. Taylor, Nucl. Phys. **B374**, 156 (1992);
F. T. Brandt and J. Frenkel, Phys. Rev. **D56**, 2453 (1997).

- [13] S. Coleman and B. Hill, Phys. Lett. **B159**, 184 (1985).
- [14] A. Das and M. Hott, Phys. Rev. **D50**, 6655 (1994).
- [15] A. Das, G. Dunne and J. Frenkel, Phys. Lett. **B472**, 332 (2000).